ON THE INTERIOR CONSTITUTION OF THE EARTH AS RESPECTS DENSITY.

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Nearly all the matter accessible to us is found to be porous. Thus the application of pressure to it tends to reduce the amount of porosity and, in consequence, augments the density of the mass. Moreover, the greater the pressure the greater is the increment of density. A familiar instance of this is the case of atmospheric air or a gas in which, provided the temperature remains constant, the density varies directly as the pressure.

It is natural to think that the matter of which the earth is composed is not excepted from this law. At small depths, it is true, the rigidity of the earth's mass interferes with its exerting any pressure, as the existence of caves shows. But at great depths where the weight of the superincumbent mass becomes very great, it is extremely probable the molecular force of cohesion gives way in a manner which allows pressure to act; which is illustrated by the behavior of ice in a glacier.

I propose to see what conclusions we are led to by adopting this relation between the density ρ and the pressure p,

$$\rho = A + Bp.$$

A and B are constants, A denoting the density at the surface, and B the rate of increase of the density per unit of pressure. In applying this formula to the atmosphere and gases, we have by Boyle's law $A = 0$. Let V denote the potential of the gravitating force of the whole mass, and let us neglect the effect of the centrifugal force arising from the rotation of the earth. Then pressure being supposed to act as though the whole mass were fluid, hydrostatics furnishes us with the equation

$$dp = \rho dV.$$

V being restricted to points on the surface or in the interior of the mass, it satisfies the partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + 4\pi\rho = 0.$$

The three equations now written may be regarded as determining the three unknowns ρ, p, and V.

By the elimination of V and p we get

$$\frac{\partial^2 \log \rho}{\partial x^2} + \frac{\partial^2 \log \rho}{\partial y^2} + \frac{\partial^2 \log \rho}{\partial z^2} + 4\pi B\rho = 0.$$

It will be seen that the constant A has disappeared from this equation. By Boyle's law in the case of gases $A = 0$; that is, the matter is capable of attenuating itself to an infinite degree, a thing very improbable. But the introduction of the constant term A, and consequent supposition of a limit to the attenuation, does not change the differential equation which ρ satisfies. This partial differential equation contains the whole theory of gases under a uniform temperature contained in vessels of any figure, and acted on by any gravitating forces; also the theory of atmospheres surrounding solid nuclei of density as heterogeneous as we please, and of any figure. The truth of the equation is not at all invalidated by any discontinuity in ρ or B; these quantities may change the law of their values as often as the problem demands.

The very simple integral of this equation in the case of the earth's atmosphere, when the attraction of the atmosphere on itself is neglected, is well known. It is our object here to examine the special solutions of this equation which are defined by the equation,

$$\rho = \text{function}\left[\sqrt{(x^2 + y^2 + z^2)}\right].$$

In this case, making $r = \sqrt{(x^2 + y^2 + z^2)}$, the partial differential equation is reduced to an ordinary one and becomes

$$\frac{d \,.\, r^2 \dfrac{d \,.\, \log \rho}{dr}}{dr} + 4\pi B r^2 \rho = 0,$$

or, as it may be written,

$$\frac{d^2 (r \log \rho)}{dr^2} + 4\pi B r \rho = 0.$$

To simplify this, let us put

$$s = 4\pi B r^2 \rho.$$

Then s being made the dependent variable, we have

$$\frac{d \,.\, r^2 \dfrac{d \,.\, \log s}{dr}}{dr} + s - 2 = 0.$$

And if $\log r = v$, it becomes

$$\frac{d^2 \log s}{dv^2} + \frac{d \,.\, \log s}{dv} + s - 2 = 0.$$

Furthermore, if $\dfrac{d \,.\, \log s}{dv} = u$, this differential equation of the first order between

u and s is obtained

$$\frac{du}{ds} = \frac{2 - (u + s)}{us}.$$

This being integrated, and u obtained in terms of s, or s in terms of u, r is given by the equation

$$r = K\epsilon^{\int \frac{ds}{us}},$$

or by the equation

$$r = K\epsilon^{\int \frac{du}{2-(u+s)}},$$

in which K is an arbitrary constant. And, if in the first of these values of $r, 4\pi Br^2\rho$ is substituted for s, the equation will be obtained which determines ρ as a function of r.

The differential equation in u and s is a particular case of the general form

$$Pdx + Qdy = 0,$$

where P and Q denote algebraical functions of x and y of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F.$$

Mathematicians have been able to obtain the integral of this, in finite terms, only when the constants A, B, etc. satisfy certain equations of condition.* Unfortunately, the differential equation under consideration does not belong to any of these particular cases. Recourse must be had to series or other methods of approximation for the determination of the relation between u and s. However, the differential equation itself will furnish the properties of the family of plane curves it defines.

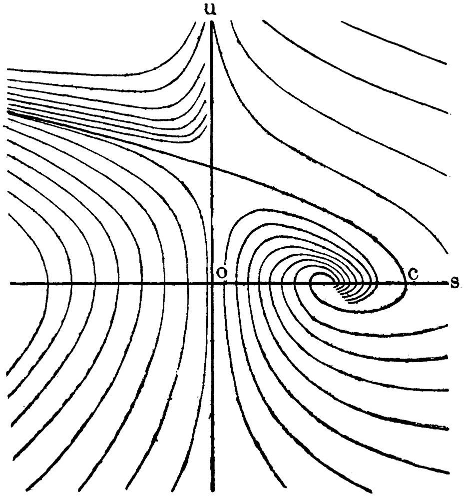
Thus u and s denoting the rectangular co-ordinates of a point in a plane, the differential equation gives immediately the means of drawing the tangent to the curve which passes through this point. Excepting at the two singular points whose co-ordinates are $u = 0$, $s = 2$ and $u = 2$, $s = 0$, for which the expression of the tangent takes the indeterminate form

$$\frac{du}{ds} = \frac{0}{0},$$

the system of curves do not intersect each other, since there is but one value of $\frac{du}{ds}$ for given values of u and s. Since the differential equation is satisfied by the condition $s = 0$, the axis of u is itself one of the system of curves, and no curve can cross it except at the point $u = 2$. If, in the differential equation, we substi-

*See Liouville, *Journal de Mathématiques*, 2e Series, Tom. III, p. 417.

tute $2 + du$ for u, and ds for s, it is clear that only one curve passes through this point, and that its tangent here is given by the equation $du/ds \doteq -\frac{1}{3}$. The axis of u, between the points $u = 2$ and $u = \infty$, is an asymptote to the whole system of curves. The axis of s is intersected at right angles by the system of curves. Investigating what occurs at the point $s = 2$ on this axis, we substitute du for u and $2 + ds$ for s, and obtain for determining du/ds at this point the following quadratic



$$\left(\frac{du}{ds}\right)^2 + \frac{1}{2}\frac{du}{ds} + \frac{1}{2} = 0,$$

the roots of which are imaginary. Hence no curve passes through this point, and it is easy to see that the system of curves makes an infinite number of turns about it.

The tangent to any curve, at its intersection with the straight line whose equation is $u + s = 2$, is parallel to the axis of s. When u and s are both very great, the tangent to the curve approximates to parallelism with the axis of s. When s is very great and u small in comparison, the differential equation becomes approximately

$$u\frac{du}{ds} = -1;$$

or integrated,

$$u^2 = 2(s_0 - s),$$

if s_0 is the value of s when $u = 0$. Hence the curves in the vicinity of the axis of s approximate to the parabola, in measure as we recede from the origin of co-ordinates.

It is very easy to draw the curves connecting all the points possessing parallel tangents. For convenience let a denote the common value of ds/du for these points; then the differential equation furnishes

$$(u + a)(s + a) = a(a + 2).$$

Thus these curves are equilateral hyperbolas having their asymptotes parallel to the axis of co-ordinates.

Thus much in regard to the properties of the curves defined by the differ-

ential equation under consideration. But, for the special physical problem we have in view, there is no necessity to attend to the course of the curves through the whole plane. The density being supposed to increase with augmentation of pressure, B is necessarily positive, and r and ρ, from the nature of the problem, being the same; s is likewise a positive quantity. There is then need only of considering the curves on the positive side of the axis of u. Moreover, since

$$u = \frac{d \,.\, \log (r^2 \rho)}{d \,.\, \log r} = \frac{r}{\rho}\frac{d\rho}{dr} + 2,$$

and $d\rho/dr$ is always negative when the force is directed towards the centre of the mass, there is no need of attending to the curves in the portion of the plane for which $u > 2$.

Before proceeding to the special problem we have in hand, I propose to illustrate the general theory by considering the density of the earth's atmosphere. It must be remembered that, in the usual manner of treating this question, the attraction of the atmosphere on itself is neglected; here, however, it is taken into account. Boyle's law being supposed to hold exactly, we shall have

$$\rho = Bp.$$

To integrate the differential equation between u and s, it will be necessary to obtain from observation the initial values of these two variables which hold at the surface of the earth. Let us denote these by u_0 and s_0; and by a similar notation the values of all the variables at the earth's surface. The values of u_0 and s_0 result from those of certain well-known physical constants.

Let $D =$ the density of mercury,
$h =$ the altitude of the barometer,
$g =$ the force of gravity,
$R =$ the mean density of the earth.

From an equation just given we have

$$u_0 = r_0 \left(\frac{d \,.\, \log \rho}{dr}\right)_0 + 2$$

$$= \frac{r_0}{p_0}\left(\frac{dp}{dr}\right)_0 + 2.$$

But we also evidently have

$$p_0 = gDh,$$

$$\left(\frac{dp}{dr}\right)_0 = -g\rho_0.$$

Substituting these values, $$u_0 = 2 - \frac{\rho_0 r_0}{Dh}.$$

Thus it is apparent that u is independent of the units assumed for the measurement of lengths and densities. In the next place

$$B = \frac{\rho_0}{p_0} = \frac{\rho_0}{gDh}.$$

But we have $$g = \frac{4\pi R r_0^3}{3} \cdot \frac{1}{r_0^2} = \tfrac{4}{3}\pi R r_0.$$

Thence we get $$s_0 = 4\pi B r_0^2 \rho_0 = \frac{3\rho_0^2 r_0}{DRh}.$$

Thus s is also independent of the just mentioned units.

Let us adopt the following values of the constants which enter into the expressions of u_0 and s_0:—

$$\begin{aligned} r_0 &= 6365419 \text{ metres,} \\ h &= 0.76 \text{ metres,} \\ \rho_0 &= 0.001293187, \\ D &= 13.596, \\ R &= 5.67. \end{aligned}$$

The value of ρ_0 is that found by Regnault* for the temperature 0° of the centigrade scale and the given altitude of the barometer; r_0 is the distance of his observatory from the centre of the earth according to Bessel's dimensions of the terrestrial spheroid; and the value of R is that determined by Baily in his repetition of the Cavendish experiment. With these data we obtain the following values of u_0 and s_0:—

$$\begin{aligned} u_0 &= -\,794.6425, \\ s_0 &= 0.5450835. \end{aligned}$$

Having these initial values we can easily integrate the differential equation connecting u and s by mechanical quadratures or series, in the direction of s diminishing until s becomes so small as to be of no account. The corresponding values of r and ρ could then be found as we have already explained. However, the differences between the numerical values obtained by this method and those resulting from neglecting the action of the atmosphere on itself would be insensible.

We pass now to the problem of the mass of the earth. Let us here denote the values of the variables which hold at the centre by the subscript ($_0$). If the

*_Mémoires de l'Académie des Sciences de Paris_, Tom. XXI.

density at the centre be finite we must have $s_0 = 0$; and the differential equation

$$\frac{ds}{du} = \frac{us}{2 - (u + s)}$$

shows that $u_0 = 2$, else s would be 0 for all values of u. Hence the curve we have to consider, in this case, is the single one which passes through the singular point $u = 2$, $s = 0$.

The mass included in the sphere whose radius is r, is

$$\begin{aligned} M &= \frac{1}{B}\int_0^r s dr \\ &= -\frac{1}{B} r^2 \frac{d \,.\, \log \rho}{dr} \\ &= \frac{1}{B} r (2 - u). \end{aligned}$$

Hence, denoting the values of the variables at the earth's surface by the subscript $(_1)$, and R denoting, as before, the mean density of the earth, we shall have

$$\frac{4\pi}{3} R r_1^3 = \frac{r_1 (2 - u_1)}{B}.$$

Whence we derive $$B = \frac{3(2 - u_1)}{4\pi R r_1^2},$$

and $$s_1 = 3(2 - u_1)\frac{\rho_1}{R}.$$

Then if we draw in the plane the right line whose equation is

$$s = 3\frac{\rho_1}{R}(2 - u),$$

the co-ordinates of its intersection with the curve defined by the differential equation and passing through the singular point $u = 2$, $s = 0$, will be the values of u_1 and s_1. This right line passes through the point $u = 2$, $s = 0$, and it is readily ascertained from the differential equation that upon this curve u constantly diminishes as s augments until it becomes 0. The lines can therefore intersect on the positive side of the axis of s only when

$$6\frac{\rho_1}{R} > OC,$$

where OC is the distance from the origin of the point where the mentioned curve crosses the axis of s.

In order to illustrate the general theory by an application, I have computed

by mechanical quadratures the values of the variable s and the function necessary for obtaining r. For this purpose it will be well to substitute for the independent variable u the variable $z = 2 - u$. The results obtained are given in the following table at intervals of 0.1 in z:—

z	s	s/z	$\int \frac{dz}{s-z}$	log r	log s/r^2
0.0	0.000	3.000	$-\infty$	$-\infty$	0.4771
0.1	0.294	2.940	− 1.1360	9.5065	0.4553
0.2	0.576	2.879	− 0.7737	9.6640	0.4323
0.3	0.846	2.818	− 0.5546	9.7592	0.4088
0.4	1.103	2.757	− 0.3938	9.8290	0.3845
0.5	1.348	2.695	− 0.2646	9.8851	0.3594
0.6	1.580	2.633	− 0.1551	9.9326	0.3333
0.7	1.799	2.570	− 0.0589	9.9744	0.3061
0.8	2.005	2.507	+ 0.0279	0.0121	0.2780
0.9	2.198	2.442	+ 0.1078	0.0468	0.2485
1.0	2.378	2.378	+ 0.1825	0.0792	0.2176
1.1	2.543	2.312	+ 0.2533	0.1100	0.1854
1.2	2.695	2.246	+ 0.3213	0.1396	0.1514
1.3	2.832	2.178	+ 0.3874	0.1682	0.1155
1.4	2.953	2.110	+ 0.4522	0.1964	0.0776
1.5	3.060	2.040	+ 0.5163	0.2242	0.0372
1.6	3.149	1.968	+ 0.5806	0.2522	9.9939
1.7	3.222	1.895	+ 0.6457	0.2804	9.9473
1.8	3.276	1.820	+ 0.7123	0.3094	9.8966
1.9	3.310	1.742	+ 0.7816	0.3394	9.8414
2.0	3.322	1.661	+ 0.8547	0.3712	9.7791
2.1	3.309	1.576	+ 0.9336	0.4055	9.7088
2.2	3.265	1.484	+ 1.0215	0.4436	9.6266
2.3	3.182	1 384	+ 1.1239	0.4881	9.5365

Let us suppose that the surface density of the earth $\rho_1 = 2.7$ and the mean density $R = 5.67$. Then at the surface of the earth the value of s/z must be

$$\frac{s_1}{z_1} = 3\frac{\rho_1}{R} = 1.4286.$$

By interpolating in the table it is found that this value corresponds to the following values of the principal variables:—

$$z = 2.257,$$
$$s = 3.224,$$
$$\log r = 0.4681,$$
$$\log \frac{s}{r^2} = 9.5722.$$

Now the last two quantities are the logarithms of the surface values of the radius and the density measured in such units as in every case will give the simplest values to the arbitrary constants. But let us take the radius at the surface as the linear unit, and represent the surface density as 2.7. Then to reduce the numbers so as to correspond to these units, it is evident we must add 9.5319 to the logarithms in the column of $\log r$, and 0.8592 to the logarithms in the column of $\log s/r^2$. Thus are obtained the following corresponding values of r and ρ:—

r	ρ	r	ρ
0.000	21.69	0.469	10.25
0.109	20.63	0.501	9.43
0.157	19.57	0.535	8.65
0.195	18.54	0.570	7.88
0.230	17.53	0.608	7.13
0.261	16.54	0.649	6.40
0.291	15.58	0.694	5.70
0.321	14.63	0.743	5.02
0.350	13.72	0.800	4.35
0.379	12.81	0.866	3.70
0.408	11.93	0.945	3.06
0.438	11.08	1.000	2.70

It will be noticed that the density at the centre is almost double of that given

by Laplace's formula; and it seems that this supposition as to the law of density will not fit the phenomena as well as the latter.

The limit beneath which the ratio ρ_1/R cannot be reduced without the problem failing to have a solution, is of interest. If the curve employed for the solution of this problem is prolonged until its tangent passes through the singular point on the axis of u, which it plainly must do before the curve crosses the axis of s a second time, this tangent affords the limit sought for the ratio $3\,\rho_1/R$. The tangents of the curves, at the points of the plane whose co-ordinates satisfy the equation

$$\frac{2-(u+s)}{us}=\frac{u-2}{s},$$

pass through the mentioned singular point. This equation in a simpler form is

$$s=(1+u)(2-u),$$

which consequently represents a parabola passing through both singular points, and having its axis parallel to that of s. By the employment of mechanical quadratures, the following additional points of the curve have been obtained:—

s	z	s	z
3.0	2.420	2.3	2.499
2.9	2.458	2.2	2.478
2.8	2.486	2.1	2.446
2.7	2.505	2.0	2.403
2.6	2.515	1.9	2.345
2.5	2.518	1.8	2.264
2.4	2.513	1.75	2.204

From these it is evident the point $u=-0.2$, $s=1.76$ which lies on the just-mentioned parabola is also very nearly on the employed curve. Hence if ρ_1/R is less than a fraction which is approximately $\frac{4}{15}$, there is no solution.

The number of solutions in any particular case is deserving of notice. The integral

$$\int\frac{dz}{s-z}$$

is proportional to the value of $\log r$. It does not become infinite until the curve has made an infinite number of turns about the singular point on the axis of s.

This may be shown by a transformation of variables. Let us adopt polar coordinates, the singular point being the pole, and thus put

$$s = \text{w} \cos \theta + 2,$$
$$z = \text{w} \sin \theta + 2.$$

The differential equation then becomes

$$\frac{d\text{w}}{\text{w}} = - \frac{\text{w} \sin \theta \cos^2 \theta + \sin^2 \theta + \sin \theta \cos \theta}{\text{w} \cos \theta \sin^2 \theta + 1 + \sin^2 \theta - \sin \theta \cos \theta} d\theta.$$

And we have

$$\int \frac{dz}{s - z} = \int \frac{d\theta}{\text{w} \cos \theta \sin^2 \theta + 1 + \sin^2 \theta - \sin \theta \cos \theta}.$$

The denominator of these expressions cannot vanish unless w exceed 2, and it is plain that it remains positive and finite for all values of θ. Thus r becomes infinite only when θ does. Consequently there are an infinite number of solutions when $\rho_1/R = \frac{1}{3}$; and a less number when ρ_1/R is either less or greater than this. With the value we have attributed to this fraction in the case of the earth, the course of the curve shows that there is but one solution.

EXERCISES.

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Derive geometrically the usual expressions for the radius of the circle inscribed in a triangle, and for the area of the triangle, in terms of the sides.

[*F. H. Loud.*]

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In any triangle ABC let a circle be inscribed touching the sides AB, BC, CA in N, L, M respectively. Let the centre O of this circle be joined to the vertices, and from O let OP, OQ be drawn perpendicular respectively to OC, OB, and cutting BC in P and Q. Then if NP and AQ be drawn, these lines will be parallel as will also AP and MQ. [*F. H. Loud.*]

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A circle cuts a parabola and the centroid of the four points of intersection is found. What is the locus of the centre of the circle if this point be fixed?

[*W. M. Thornton.*]